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# INVARIANT IMBEDDING AND OPTIMUM BEAM DESIGN WITH DISPLACEMENT CONSTRAINTS

NÉSTOR DISTÉFANO and RICARDO TODESCHINI

University of California, Berkeley, California 94720

**Abstract**—In the present paper, the condition of optimality for a beam under elastic foundation subject to a displacement constraint are derived from the calculus of variations and subsequently employed as the starting point in the development of a stable method for the numerical solution of the optimization problem. Using ideas of invariant imbedding and the method of successive approximations, the pertinent nonlinear boundary value problem is reduced to a two sweep iterative procedure in terms of a system of Riccati differential equations subject to initial values, exhibiting favorable stability properties. Two examples thoroughly developed, are finally presented to illustrate the application and the accuracy of the method.

## 1. INTRODUCTION

A CLASS of well posed problems in beam optimization may be formulated by requiring minimum volume subject to appropriate constraints on the displacement field. A number of papers have been devoted to this problem [1–3]. While in those papers the derivation of pertinent conditions of optimality have been mainly emphasized, a systematic treatment of the solution of the resulting nonlinear boundary value problem appears to have been neglected in the literature. Interest in this direction of research stems from obvious practical considerations in addition to some fundamental ones. In effect, the nonlinear-boundary-value character of the necessary conditions prevents in general a direct treatment of the equations by means of standard numerical techniques. Thus the interest to reduce this problem to alternative initial-value formulations for which many standard numerical procedures are available. This problem has been extensively studied in connection with optimal control theory. For example, see [4] for references. Unfortunately, the methods of control theory cannot in general be applied directly to the present type of problems, mainly because of the different character of the equations involved. While in the theory of control of dynamical systems the equations of evolution are initial valued, in structural mechanics we usually deal with spatial descriptions in terms of differential equations subject to boundary conditions. It seems therefore, that future research efforts in this direction should recognize this fact, particularly in those problems associated with the stability of the resulting numerical algorithms.

In the present paper we use the calculus of variations in the fashion of Pontryagin's minimum principle to derive optimality conditions for a beam under elastic foundation subject to a displacement constraint. We present this problem as representative of a larger class of problems in structural mechanics rather than one of particular importance on its own. Using ideas of invariant imbedding and the method of successive approximations, the pertinent nonlinear-mixed-boundary-value problem is reduced to a two sweep iterative procedure in terms of a system of Riccati differential equations subject to initial values and exhibiting favorable numerical stability properties. Two examples, thoroughly developed, are finally presented to illustrate the application and the accuracy of the method.

## 2. FORMULATION OF THE PROBLEM—NECESSARY CONDITIONS

We consider a beam of length $L$ supported on elastic foundation with coefficient $k(x)$ and subject to external forces $q(x)$. If $u(x)$ denotes the deflection, $v(x)$ the slope, $m(x)$ the bending moment and $t(x)$ the shear, the constitutive equations of an Euler–Bernoulli beam are given by

$$\frac{du}{dx} = v, \qquad \frac{dv}{dx} = -\frac{1}{\alpha}m, \tag{1}$$

and the equilibrium equations by

$$\frac{dm}{dx} = -t, \qquad \frac{dt}{dx} = q-ku, \tag{2}$$

where

$$\alpha = EI, \tag{3}$$

is the stiffness, $I$ the moment of inertia and $E$ is Young's modulus. We choose the stiffness $\alpha$ as the design variable and assume that the area $A$ of the cross-section of the beam is given by the formula

$$A(x) = g(\alpha, x), \tag{4}$$

where $g$ is a function that depends on the particular geometry of the cross section. For example, for rectangular beams with constant width $b$ and variable height $h$, $g = (12b^2\alpha/E)^{1/3}$. The volume of the beam is given by

$$V = \int_0^L g(\alpha, x)\,dx. \tag{5}$$

Considering $u$, $v$, $m$ and $t$ subject to an appropriate set of boundary conditions, we can now formulate the following minimum volume problem for a prescribed displacement $u_1$ at $x = x_1$ and additional inequality constraints:

Minimize the quantity $V$ given by (5) subject to the conditions

$$u(x_1) = u_1, \tag{6}$$

$$\varphi_0(\alpha, x) \leq 0, \tag{7}$$

and possibly to a number of additional constraints

$$\varphi_i(m, t, \alpha, x) \leq 0, \qquad i = 1, 2, \ldots \tag{8}$$

We assume that this optimization problem is well posed, i.e. a solution exists, is unique and continuous with respect to boundary conditions and constraints. The study of classes of constraints under which a given optimization problem is well posed, is one of great interest and importance, but it is beyond the scope of this paper whose main purpose is the discussion of numerical computational procedures.

The derivation of necessary conditions for this problem can be done using the calculus of variations. In order to incorporate the constraint on the deflection $u$, instead of (6) we consider the equivalent integral expression

$$\int_0^L u(x)\delta(x-x_1)\,\mathrm{d}x = u_1, \tag{9}$$

where $\delta$ is the Dirac delta. Now we form the Hamiltonian

$$\mathscr{H} = g(\alpha, x)+\lambda u\delta(x-x_1)+\lambda_1 v-\lambda_2\frac{1}{\alpha}m-\lambda_3 t+\lambda_4(q-ku), \tag{10}$$

where $\lambda$ and $\lambda_1$–$\lambda_4$ are Lagrange multipliers used to incorporate the constraints (1), (2) and (9). It is well known (see for example [5]) that the multipliers must satisfy the adjoint differential equations

$$\begin{aligned}
\frac{\mathrm{d}\lambda}{\mathrm{d}x} &= -\frac{\partial\mathscr{H}}{\partial u_1} = 0,\\
\frac{\mathrm{d}\lambda_1}{\mathrm{d}x} &= -\frac{\partial\mathscr{H}}{\partial u} = k\lambda_4-\lambda\delta(x-x_1),\\
\frac{\mathrm{d}\lambda_2}{\mathrm{d}x} &= -\frac{\partial\mathscr{H}}{\partial v} = -\lambda_1,\\
\frac{\mathrm{d}\lambda_3}{\mathrm{d}x} &= -\frac{\partial\mathscr{H}}{\partial m} = \frac{1}{\alpha}\lambda_2,\\
\frac{\mathrm{d}\lambda_4}{\mathrm{d}x} &= -\frac{\partial\mathscr{H}}{\partial t} = \lambda_3.
\end{aligned} \tag{11}$$

From the first equation (11) we see that $\lambda$ is a constant, to be chosen such as to enforce condition (6). Comparison of (1) and (2) with equations (11) shows that the Lagrange multipliers $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_4$ are the forces and displacements of the beam subject to a virtual concentrated load of intensity $\lambda$ at $x = x_1$. More precisely,

$$\begin{aligned}
\lambda_4 &= \lambda\bar{u},\\
\lambda_3 &= \lambda\bar{v},\\
\lambda_2 &= -\lambda\bar{m},\\
\lambda_1 &= -\lambda\bar{t},
\end{aligned} \tag{12}$$

where $\bar{u}$, $\bar{v}$, $\bar{m}$ and $\bar{t}$ are the displacements and forces due to a unit virtual force at $x_1$. Minimization of (10), taking into account (12), yields the optimality condition

$$\alpha_{\mathrm{opt}} = \arg\min_{\alpha}\left[\lambda\frac{m\bar{m}}{\alpha}+g(\alpha, x)\right], \tag{13}$$

where $\alpha_{opt}$ denotes the optimum design. It can be proved that (13) furnishes a sufficient condition for a relative minimum, under appropriate convexity conditions on function $g(\alpha, x)$.

## 3. AN ALTERNATIVE DERIVATION OF THE OPTIMALITY CONDITION

Instead of incorporating the differential constraints (1) and (2) using the Lagrange multipliers $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_4$, we can proceed in the following way. In place of (9) we use the integral representation

$$u_1 = \int_0^L \left(\frac{m\bar{m}}{\alpha} + ku\bar{u}\right) dx, \tag{14}$$

furnished by the theorem of virtual work, where $\bar{m}$ and $\bar{u}$ are the moment and the displacement, respectively, of the beam subject to a virtual load applied at $x_1$ and in the direction of the prescribed displacement. Combining (5) and (14) we form the Hamiltonian

$$\mathscr{H}_1 = g(\alpha, x) + \lambda\left(\frac{m\bar{m}}{\alpha} + ku\bar{u}\right), \tag{15}$$

where $\lambda \geq 0$ is a Lagrange multiplier that satisfies the equation

$$\frac{d\lambda}{dx} = -\frac{\partial \mathscr{H}_1}{\partial u_1} = 0. \tag{16}$$

Hence $\lambda$ is a positive constant to be chosen such as to satisfy (6). Clearly, minimization of (15) yields (13), as expected.

## 4. OPTIMUM CANTILEVER ON ELASTIC FOUNDATION FOR PRESCRIBED DISPLACEMENT

To illustrate the method and the problems associated with the numerical computations, we consider the case of a cantilever beam laying on elastic foundation and subject to a prescribed displacement at $x = x_1$. The boundary conditions of equations (1) and (2) are

$$\begin{aligned} u(0) &= 0, \qquad m(L) = M, \\ v(0) &= 0, \qquad t(L) = T. \end{aligned} \tag{17}$$

For simplicity we rewrite equations (11) in terms of $\bar{u}$, $\bar{v}$, $\bar{m}$ and $\bar{t}$ given by (12), i.e.

$$\begin{aligned} \frac{d\bar{u}}{dx} &= \bar{v}, \\ \frac{d\bar{v}}{dx} &= -\frac{1}{\alpha}\bar{m}, \\ \frac{d\bar{m}}{dx} &= -\bar{t}, \\ \frac{d\bar{t}}{dx} &= \delta(x - x_1) - k\bar{u}, \end{aligned} \tag{18}$$

subject to the homogeneous boundary conditions

$$\bar{u}(0) = \bar{v}(0) = 0$$
$$\bar{m}(L) = \bar{t}(L) = 0. \tag{19}$$

So formulated the solution of the optimum beam reduces to the task of integrating equations (1), (2) and (18) subject to boundary conditions (17) and (19), respectively. These two systems of equations are coupled together through the optimality condition (13). In a number of important cases in the applications we possess an explicit representation for the minimum operation in (13), simplifying in some sense the solution of the system. In any case, this nonlinear boundary value problem can be integrated using a quasilinearization scheme as in [6]. We shall not pursue this path here, i.e. a direct treatment of the nonlinear boundary value problem, in favor of the implementation of a simple, first order, stable iterative method based in ideas of invariant imbedding. This is done in the next section.

## 5. INVARIANT IMBEDDING

For a given nominal design $\alpha$, we consider the uncoupled linear boundary-value systems given by equations (1), (2), (17) and (18), (19), respectively. First we consider system (1), (2). Instead of boundary conditions (17) we set

$$u(X) = w, \qquad m(L) = M,$$
$$v(X) = z, \qquad t(L) = T, \tag{20}$$

i.e. we consider the families of beams of length $L-X$ subject to arbitrary displacements $w$ and $z$ at the end $x = X$. This is an "imbedding" procedure in the fashion of invariant imbedding [7], that affords the property of reducing the computation of the original boundary-value problem (1), (2), (17), to a stable, two-sweep, procedure.

We seek solutions of the form

$$t(X) = r_1(X)w + r_{12}(X)z + s_1(X),$$
$$m(X) = r_{21}(X)w + r_2(X)z + s_2(X). \tag{21}$$

Differentiation of (21) with respect to $X$ and elimination of the derivatives $w' = u'(X)$ and $z' = v'(X)$ using equations (1) and (2) evaluated at $x = X$ and further elimination of $t(X)$ and $m(X)$ using (21) yields

$$\left(r_1' + k - r_{12}\frac{1}{\alpha}r_{21}\right)w + \left(r_{12}' - r_{12}\frac{1}{\alpha}r_2\right)z + \left(s_1' - q - r_{12}\frac{1}{\alpha}s_2\right) = 0,$$

and

$$\left(r_{21}' + r_1 - r_2\frac{1}{\alpha}r_{21}\right)w + \left(r_2' + r_{12} + r_{21} - r_2\frac{1}{\alpha}r_2\right)z + \left(s_2' + s_1 - r_2\frac{1}{\alpha}s_2\right) = 0,$$

a system of equations that must be valid for any $w$ and $z$. Therefore

$$
\begin{aligned}
r'_1 &= -k + r_{12}\frac{1}{\alpha}r_{21},\\
r'_{12} &= -r_1 + r_{12}\frac{1}{\alpha}r_2,\\
r'_{21} &= -r_1 + r_2\frac{1}{\alpha}r_{21},\\
r'_2 &= -r_{12} - r_{21} + r_2\frac{1}{\alpha}r_2,\\
s'_1 &= q + r_{12}\frac{1}{\alpha}s_2,\\
s'_2 &= -s_1 + r_2\frac{1}{\alpha}s_2,
\end{aligned}
\tag{22}
$$

a system of Riccati equations subject to the initial conditions at $X = L$,

$$
\begin{aligned}
r_1(L) = r_{12}(L) = r_{21}(L) = r_2(L) = 0,\\
s_1(L) = T, \qquad s_2(L) = M,
\end{aligned}
\tag{23}
$$

obtained upon consideration of equations (20) and (21). Consideration of the second and third equation in (22), and corresponding initial conditions readily yields

$$
r_{12} = r_{21}, \tag{24}
$$

an expression of Maxwell theorem derived from invariant imbedding. Therefore equations (22) and (23) reduce to

$$
\begin{aligned}
r'_1 &= -k + \frac{1}{\alpha}r_{12}^2, & r_1(L) &= 0,\\
r'_{12} &= -r_1 + \frac{1}{\alpha}r_{12}r_2, & r_{12}(L) &= 0,\\
r'_2 &= -2r_{12} + \frac{1}{\alpha}r_2^2, & r_2(L) &= 0,\\
s'_1 &= q + \frac{1}{\alpha}r_{12}s_2, & s_1(L) &= T,\\
s'_2 &= -s_1 + \frac{1}{\alpha}r_2 s_2, & s_2(L) &= M.
\end{aligned}
\tag{25}
$$

Substitution of $m(X)$ given by the second equation (21) in equations (1) evaluated at $x = X$, and due consideration of equations (17) and (20), yield

$$
\begin{aligned}
\frac{du}{dX} &= v, & u(0) &= 0,\\
\frac{dv}{dX} &= -\frac{1}{\alpha}(r_{12}u + r_2 v + s_2), & v(0) &= 0,
\end{aligned}
\tag{26}
$$

an initial value problem in the forward direction for the deflection $u$ and slope $v$ of the beam, where the quantities $r_{12}$, $r_2$ and $s_2$ are given by the integration of (25) in the backwards direction. Substitution of $u$ and $v$ given by (26) in (21), taking into account that $w = u(X)$ and $z = v(X)$, finally yields the remaining state variables of the beam i.e. the moment $m$ and the shear force $t$ in the interval $0 \le X \le L$.

We can treat the virtual system (18) in a similar fashion. We consider $\bar{u}$, $\bar{v}$, $\bar{m}$ and $\bar{t}$ satisfying equations (18) to be subject to the boundary conditions

$$\begin{aligned} \bar{u}(X) &= \bar{w}, & \bar{m}(L) &= 0, \\ \bar{v}(X) &= \bar{z}, & \bar{t}(L) &= 0, \end{aligned} \tag{27}$$

and write for $\bar{t}(X)$ and $\bar{m}(X)$, the missing boundary conditions of the imbedded beam of length $L-X$, equations similar to (21), i.e.

$$\begin{aligned} \bar{t}(X) &= \bar{r}_1(X)\bar{w}+\bar{r}_{12}(X)\bar{z}+\bar{s}_1(X), \\ \bar{m}(X) &= \bar{r}_{21}(X)\bar{w}+\bar{r}_2(X)\bar{z}+\bar{s}_2(X). \end{aligned} \tag{28}$$

Carrying out the same perturbation analysis done to equations (21), on equations (28), we obtain

$$\begin{aligned} \bar{r}_1' &= -k+\frac{1}{\alpha}\bar{r}_{12}^2, & \bar{r}_1(L) &= 0, \\ \bar{r}_{12}' &= -\bar{r}_1+\frac{1}{\alpha}\bar{r}_{12}\bar{r}_2, & \bar{r}_{12}(L) &= 0, \\ \bar{r}_2' &= -2\bar{r}_{12}+\frac{1}{\alpha}\bar{r}_2^2, & \bar{r}_2(L) &= 0, \\ \bar{s}_1' &= \delta(X-x_1)+\frac{1}{\alpha}\bar{r}_{12}\bar{s}_2, & \bar{s}_1(L) &= 0, \\ \bar{s}_2' &= -\bar{s}_1+\frac{1}{\alpha}\bar{r}_2\bar{s}_2, & \bar{s}_2(L) &= 0. \end{aligned} \tag{29}$$

Similarly, we can write

$$\begin{aligned} \frac{\mathrm{d}\bar{u}}{\mathrm{d}X} &= \bar{v}, & \bar{u}(0) &= 0, \\ \frac{\mathrm{d}\bar{v}}{\mathrm{d}X} &= -\frac{1}{\alpha}(\bar{r}_{12}\bar{u}+\bar{r}_2\bar{v}+\bar{s}_2), & \bar{v}(0) &= 0, \end{aligned} \tag{30}$$

an initial value problem for $\bar{u}$ and $\bar{v}$ obtained after substitution of $\bar{m}$ given by the second equation (28), in the first two equations (18) evaluated at $x = X$. Finally, $\bar{t}$ and $\bar{m}$ can be obtained from (28) recalling that $\bar{w} = \bar{u}(X)$ and that $\bar{z} = \bar{v}(X)$.

## 6. A TWO SWEEP ITERATIVE PROCEDURE

Using the results of the last section we can now implement a simple iterative method for the solution of the optimization problem. To this end let $a^{(n)}$ denote the value of the quantity $a$ at the $n$th iteration. Then at a generic iteration $(n+1)$ we use the currently available design $\alpha^{(n)}$ to integrate (25) in the backwards direction and subsequently (26) in the forward direction, computing in turn $m^{(n+1)}$ given by (21). Similarly we compute the values of $\overline{m}^{(n+1)}$ by using the two sweep process given by (28)–(30). The improved value of the design $\alpha^{(n+1)}$ follows from the optimality condition

$$\alpha^{(n+1)} = \arg\min_{\alpha}\left[\lambda^{(n+1)}\frac{m^{(n+1)}\overline{m}^{(n+1)}}{\alpha}+g(\alpha, x)\right], \tag{31}$$

where the new estimate of $\lambda = \lambda^{(n+1)}$ needs to be taken such as to ensure an appropriate convergence of the sequence $u^{(n)}(x_1)$ to the prescribed value $u_1$. This can be done in a number of ways. In general we shall use formulas of the type

$$\lambda^{(n+1)} = F(\lambda^{(n)}, u^{(n)}(x_1), u^{(n-1)}(x_1)\ldots). \tag{32}$$

For example, we could simply take

$$\lambda^{(n+1)} = \lambda^{(n)}u^{(n)}(x_1)/u_1. \tag{33}$$

The procedure is repeated until convergence is achieved. Initially we need an *a priori* estimate of the design $\alpha^{(0)}$. In the absence of special information, a uniform design $\alpha^{(0)}(x) =$ const. is usually taken as the initial design. A numerical example in Section 8 will illustrate the application of the method.

## 7. OWN WEIGHT

When the own weight of the beam is to be taken into account, $q$ in equations (2), (10) and (25) must be replaced by

$$q = p+\gamma A, \tag{34}$$

where $p$ are the external forces, $\gamma$ is the specific weight of the beam and $A$ the cross sectional area given by equation (4). Therefore the optimality condition (13) must be substituted by

$$\alpha_{\text{opt}} = \arg\min_{\alpha}\left[\lambda\frac{m\overline{m}}{\alpha}+(1+\gamma\lambda\bar{u})g(\alpha, x)\right], \tag{35}$$

and equation (34) must be taken into account when integrating equations (25). Similarly, other mass forces can be taken into account.

## 8. STABILITY CONSIDERATIONS

In order to prove the stability of the method, we should show that $u$ and $v$ given by (25) and (26) are stable with respect to small errors introduced in the computational process. To this end it is enough to prove the stability of the differential system (25) and, in particular, of the quantities $r_1$, $r_{12}$ and $r_2$. We rewrite the first three equations (25) in the matrix form

$$R' = -A-BR-RB^T+RCR, \qquad R(L) = 0, \tag{36}$$

where

$$R = \begin{vmatrix} r_1 & r_{12} \\ r_{21} & r_2 \end{vmatrix}, \quad A = \begin{vmatrix} k & 0 \\ 0 & 0 \end{vmatrix}, \quad B = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, \quad C = \begin{vmatrix} 0 & 0 \\ 0 & \frac{1}{\alpha} \end{vmatrix}.$$

The stability of the Riccati equation (36) in the backwards direction follows from the fact that $C > 0$ (since $\alpha \geq 0$). The stability of $R$ is clearly enough to ensure the stability of $u$, $v$, $m$ and $t$. Similar considerations hold for $\bar{u}$, $\bar{v}$, $\bar{m}$ and $\bar{t}$.

## 9. NUMERICAL EXAMPLES

(a) *Cantilever on elastic foundation*

We consider a beam of the sandwich type such as that whose cross-section is indicated in Fig. 1. In this case if $2h$ is a fixed quantity denoting the distance between the covering sheets and $A/2$ is the area of each one of the sheets, we have $\alpha = EI = EAh^2$ and $g$ in (4) reduces to

$$g(\alpha, x) = \alpha/Eh^2. \tag{37}$$

The volume is therefore proportional to $\alpha$, i.e.

$$V = \frac{1}{Eh^2}\int_0^L \alpha \, \mathrm{d}x. \tag{38}$$

In addition we shall take $x_1 = L$ and $q = P\delta(x-L)$, i.e. we prescribe the displacement $u_1$ in correspondence with a concentrated load applied at the free end of the cantilever. The reason to consider this simplified example is because under the present assumptions we can afford a closed form solution of the problem that can be used to compare the accuracy of the numerical procedures. In fact, in our present case we have $m = P\bar{m}$ and equation (13) reduces to

$$\alpha_{\text{opt}} = \arg\min_{\alpha}\left(\mu\frac{m^2}{\alpha}+\alpha\right), \tag{39}$$

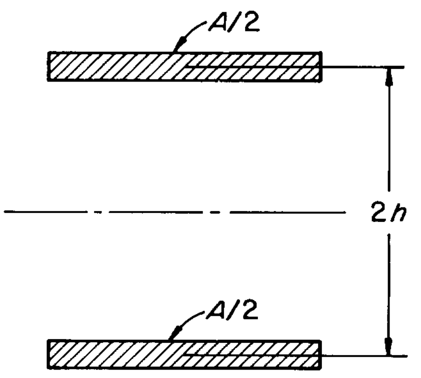


FIG. 1. Beam cross-section.

where $\mu = (Eh^2/P)\lambda$ is a constant to be determined such that $u(L) = u_1$. In the portions of the beam where the design is continuously differentiable, (39) reduces to

$$\alpha_{\text{opt}}^2 = \mu m^2. \tag{40}$$

Combining (1) and (39) we obtain

$$(u'')^2 = 1/\mu. \tag{41}$$

A solution that satisfies the nonlinear differential equation (41) and the boundary conditions $u(0) = u'(0) = 0$ and $u(L) = u_1$ is given by

$$u(x) = u_1 x^2/L^2. \tag{42}$$

Equation (42) is valid if sign $m$ = sign $u''$, a condition that is clearly fulfilled in our example. The curvature is given by

$$|u''| = (1/\mu)^{\frac{1}{2}} = 2u_1/L^2. \tag{43}$$

Combining (40) and (43),

$$\alpha_{\text{opt}} = L^2 m/2u_1 \tag{44}$$

where the bending moment $m$ can be readily computed by direct integration, namely,

$$m = P(L-x) - \int_0^x (z-x)ku\,\mathrm{d}z = PL(1-\eta) - ku_1L^2(3-4\eta+\eta^4)/12, \tag{45}$$

where we put $\eta = x/L$. In a similar fashion we can derive for the shear force $t$ the expression

$$t = P[1-4\beta(1-\eta^3)], \tag{46}$$

where

$$\beta = ku_1L/12P. \tag{47}$$

Combination of (44) and (45) yields

$$\bar{\alpha}_{\text{opt}} = 1-\eta-\beta(3-4\eta+\eta^4), \tag{48}$$

where $\bar{\alpha}$ is a dimensionless design given by

$$\bar{\alpha} = 2u_1\alpha/PL^3. \tag{49}$$

Clearly, the range of $\beta$ for which $\bar{\alpha}_{\text{opt}}$ is positive is $0 \leq \beta \leq \frac{1}{3}$. In Fig. 2, $\bar{\alpha}_{\text{opt}}$ given by (48) is presented for several values of $\beta$. For purposes of comparison, the same example was solved using the procedure outlined in Section 6. At a generic iteration we integrate equations (25) in the backwards direction using the currently available estimate for the design. There is no need to integrate equations (29) since in this particular case we have $m = P\bar{m}$. Subsequently, using the values of the $r$'s previously computed, we integrate (26) in the forward direction. In turn we calculate a new, upgraded, estimate of the design by means of

$$\alpha^{(n+1)} = (\mu^{(n+1)})^{\frac{1}{2}} m^{(n+1)}, \tag{50}$$

an expression for $\alpha^{(n+1)}$ that follows from (31) by taking derivatives and making $m = P\bar{m}$ and $g = \alpha/Eh^2$. In equation (50), $\mu^{(n+1)}$ given by

$$\mu^{(n+1)} = \frac{Eh^2}{P}\lambda^{(n+1)} \tag{51}$$

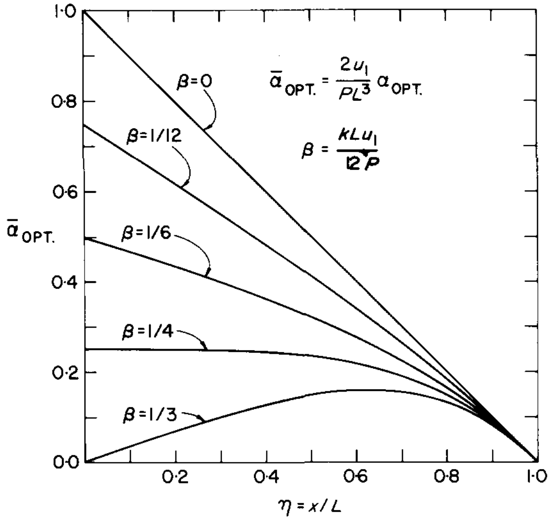


FIG. 2. Optimum beam designs.

is a constant at each iteration that must be chosen such as to ensure the convergence of the sequence $u^{(n)}(x_1)$ to the prescribed value $u_1$. We have taken in the present case

$$\mu^{(n+1)} = \mu^{(n)}u^{(n)}(x_1)/u_1. \tag{52}$$

The procedure is continued until the following convergence criterion

$$E_k = \max(|\varepsilon_k|, |\varepsilon_{k+1}|, |\varepsilon_{k+2}|) \leq 10^{-3}, \tag{53}$$

where

$$\varepsilon_k = 1 - \bar{\alpha}^{(k)}/\bar{\alpha}^{(k+1)}, \tag{54}$$

is fulfilled. All the integrations were performed numerically using an Adams Moulton scheme with step size 0·005 on a CDC 6400 computer. The resulting values for three different values of $\beta$ and at four equidistant sections of the cantilever are presented in Table 1. It is seen that in order to reach the same accuracy, the number of iterations increases for increasing values of the design parameter $\beta$. On the other hand it is of interest to note that the convergence of the process is of an oscillatory type. This can be appreciated in Fig. 3 where the relative error $\varepsilon_k$ *vs.* the number of iterations $k$ for two different values of $\beta$ has been plotted. Methods to improve the convergence properties of the process can be devised for the present problem but we shall not enter in that discussion here.

A further application of the method in connection with piecewise constant design is presented below. The design is taken of the form

$$\alpha = \sum_{i=1}^{N} c_i H(x-x_i), \qquad H(x-x_i) = \begin{cases} 1 & \text{if } x \geq x_i \\ 0 & \text{if } x < x_i. \end{cases} \tag{55}$$

The problem consists in determining the cross sectional stiffness $c_i$ and the lengths $x_i$ such as to minimize the volume of the beam. The values of $c_i$ are constrained to be one of any possible combinations of a given set of values $\alpha_1, \alpha_2, \ldots, \alpha_M$. This is a version of a piecewise constant optimum design problem that occurs when the flanges of the beam

TABLE 1. COMPARISON OF EXACT AND COMPUTED DESIGNS (CANTILEVER BEAM ON ELASTIC FOUNDATION)

| $\beta$ | $\eta$ | $\bar{\alpha}(\eta)$ exact | $\bar{\alpha}^{(0)}$ initial estimate | $\bar{\alpha}^{(k)}(\eta)$ | $\bar{\alpha}^{(k+1)}(\eta)$ | $E_k = \max(\lvert\varepsilon_k\rvert, \lvert\varepsilon_{k+1}\rvert, \lvert\varepsilon_{k+2}\rvert)$ | $\lvert\bar{\alpha} - \bar{\alpha}^{(k)}\rvert$ |
|---|---|---|---|---|---|---|---|
| | 0·00 | 0·75000 | 2·0 | 0·74922(8) | 0·74983(9) | $0{\cdot}82 \times 10^{-3}$(8) | $0{\cdot}78 \times 10^{-3}$(8) |
| $\frac{1}{12}$ | 0·25 | 0·58301 | 2·0 | 0·58246(8) | 0·58290(9) | $0{\cdot}75 \times 10^{-3}$(8) | $0{\cdot}55 \times 10^{-3}$(8) |
| | 0·50 | 0·41146 | 2·0 | 0·41114(8) | 0·41140(9) | $0{\cdot}64 \times 10^{-3}$(8) | $0{\cdot}32 \times 10^{-3}$(8) |
| | 0·75 | 0·22363 | 2·0 | 0·22351(8) | 0·22362(9) | $0{\cdot}48 \times 10^{-3}$(8) | $0{\cdot}12 \times 10^{-3}$(8) |
| | 0·00 | 0·50000 | 2·0 | 0·50009(15) | 0·49979(16) | $0{\cdot}60 \times 10^{-3}$(15) | $0{\cdot}09 \times 10^{-3}$(15) |
| $\frac{1}{6}$ | 0·25 | 0·41602 | 2·0 | 0·41604(15) | 0·41584(16) | $0{\cdot}48 \times 10^{-3}$(15) | $0{\cdot}02 \times 10^{-3}$(15) |
| | 0·50 | 0·32292 | 2·0 | 0·32315(14) | 0·32288(15) | $0{\cdot}83 \times 10^{-3}$(14) | $0{\cdot}23 \times 10^{-3}$(14) |
| | 0·75 | 0·19727 | 2·0 | 0·19732(14) | 0·19721(15) | $0{\cdot}55 \times 10^{-3}$(14) | $0{\cdot}05 \times 10^{-3}$(14) |
| | 0·00 | 0·25000 | 2·0 | 0·25026(28) | 0·25040(29) | $0{\cdot}59 \times 10^{-3}$(28) | $0{\cdot}26 \times 10^{-3}$(28) |
| $\frac{1}{4}$ | 0·25 | 0·24902 | 2·0 | 0·24907(27) | 0·24927(28) | $0{\cdot}80 \times 10^{-3}$(27) | $0{\cdot}05 \times 10^{-3}$(27) |
| | 0·50 | 0·23428 | 2·0 | 0·23432(26) | 0·23450(27) | $0{\cdot}80 \times 10^{-3}$(26) | $0{\cdot}06 \times 10^{-3}$(26) |
| | 0·75 | 0·17090 | 2·0 | 0·17084(25) | 0·17097(26) | $0{\cdot}74 \times 10^{-}3$(25) | $0{\cdot}06 \times 10^{-3}$(25) |

$k$ = iteration number.

must be constructed using a number of available sections $\alpha_1, \alpha_2, \ldots, \alpha_M$. Substituting $\alpha$ given by (55) in the optimality condition (39), we can solve this problem by minimizing with respect to all possible values of $c_i$. In the present example we have considered $\alpha_1 = \alpha_2 = \ldots = \alpha_6 = 0{\cdot}125$. Therefore $c_i = 0{\cdot}125j(i)$ where the possible values of $j$ are 1–6. The results are shown in Fig. 4 where a comparison with the unconstrained solution is possible.

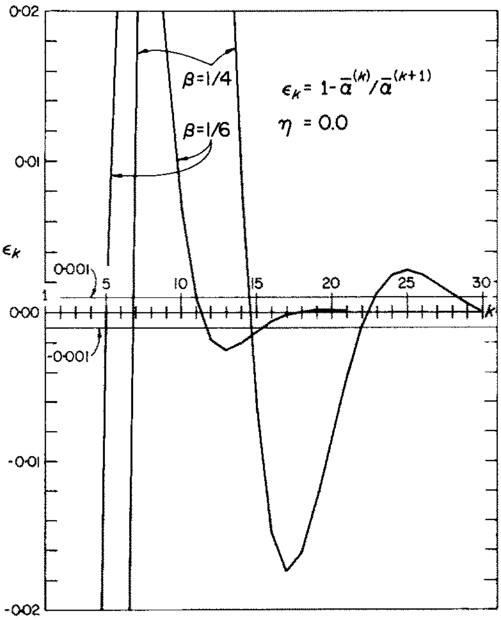


FIG. 3. Relative error vs. number of iterations.

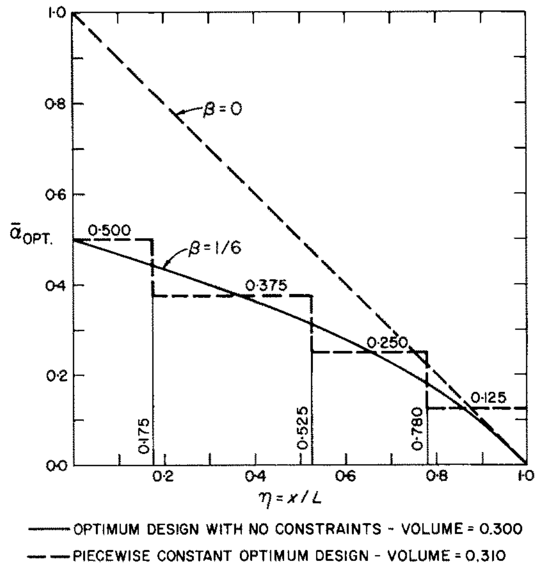


FIG. 4. Comparison of design criteria.

(b) *Clamped beam under uniformly distributed load*

An optimal design is characterized by

$$m\bar{m} > 0, \tag{56}$$

a convexity condition that follows from consideration of the optimality condition (13). During the computation of the successive approximations following the procedure developed in Section 6, equation (56) might be violated leading to an indeterminacy in equation (13), unless additional information is furnished. This can be done in a number of ways. For example, we can require a positive lower bound for the design $\alpha$, i.e.

$$\alpha \geq \delta > 0, \tag{57}$$

where $\delta$ is usually taken of the order of magnitude of the step of integration of the equations. This problem did not occur in our previous example involving the cantilever on elastic foundation, because in that case, equation (56) was identically satisfied since $m = \bar{P}m$.

The purpose of the present example is to show that equation (57) is enough to bypass the difficulties created by a possible violation of (56) during the first iterations of the process. To this end we consider a sandwich clamped beam of length $2L$ subject to uniformly distributed load $q$. It is required that the deflection at the middle be $u_1$. The exact optimal solution can be obtained without difficulties. In fact, assuming $\alpha$ to be continuously differentiable, the optimality condition is given by

$$\alpha_{\text{opt}} = \mu^{\frac{1}{2}}(m\bar{m})^{\frac{1}{2}}, \tag{58}$$

where $\mu^{\frac{1}{2}}$ is a positive constant to be determined such as to satisfy the deflection condition at the center of the beam. Since the beam is clamped at both ends, there will be an inflection point symmetrically located at a distance $X$ from both ends. The value of $X$ will be obtained from the condition of minimum volume of the beam, which on account of (58) reads

$$\min_{X} \int_0^L (m\bar{m})^{\frac{1}{2}}\, dx, \tag{59}$$

where $m$ and $\bar{m}$ are given by

$$\begin{aligned} m &= -(X-x)(2L-X-x)q/2, \\ \bar{m} &= -(X-x)/2. \end{aligned} \tag{60}$$

From this minimization problem we obtain $X$ as the real solution of

$$(512-224\sqrt{2})X^3-(1473-672\sqrt{2})LX^2+(1395-672\sqrt{2})L^2X-(433-224\sqrt{2})L^3 = 0, \tag{61}$$

or, $X = 0{\cdot}5022$, with four exact decimal places. The deflection $u$ can be readily obtained by integration of the differential equations

$$\begin{aligned} u'' &= \left(\frac{q}{\mu}\right)^{\frac{1}{2}}(2L-X-x)^{\frac{1}{2}}, \qquad 0 \le x \le X, \\ u'' &= -\left(\frac{q}{\mu}\right)^{\frac{1}{2}}(2L-X-x)^{\frac{1}{2}}, \qquad X < x \le L, \end{aligned} \tag{62}$$

subject to the initial conditions

$$u(0) = u'(0) = 0,$$

and the continuity conditions

$$u(X^-) = u(X^+), \qquad u'(L) = 0.$$

Making

$$\mu^{\frac{1}{2}} = \frac{2}{15}\frac{q^{\frac{1}{2}}}{u_1}[(16\sqrt{2}-7)(L-X)^{\frac{5}{2}}-(4L-7X)(2L-X)^{\frac{3}{2}}], \tag{63}$$

we satisfy the deflection condition at $x = L$.

The optimal design obtained using the method of successive approximations may be compared with the exact one given by (58) and (63) in Fig. 5 and Table 2. In that example, the dimensionless design $\bar{\alpha} = (10u_1/qL^4)\alpha$ and the dimensionless quantities $\eta = x/L$ and $\eta_0 = X/L$ have been introduced. The procedure used to compute the solution using the

TABLE 2. COMPARISON OF EXACT AND COMPUTED DESIGNS (CLAMPED–CLAMPED BEAM UNDER UNIFORMLY DISTRIBUTED LOAD)

| $\eta$ | $\bar{\alpha}(\eta)$ exact | $\bar{\alpha}^{(0)}(\eta)$ initial approximate | $\bar{\alpha}^{(1)}(\eta)$ | $\bar{\alpha}^{(2)}(\eta)$ | $\bar{\alpha}^{(10)}(\eta)$ | $\bar{\alpha}^{(20)}(\eta)$ | $\bar{\alpha}^{(30)}(\eta)$ |
|---|---|---|---|---|---|---|---|
| 0·00 | 0·75560 | 0·5 | 0·58926 | 0·74725 | 0·72994 | 0·74267 | 0·74948 |
| 0·20 | 0·42326 | 0·5 | 0·30957 | 0·39508 | 0·39632 | 0·40962 | 0·41649 |
| 0·40 | 0·13168 | 0·5 | 0·05271 | 0·08591 | 0·10335 | 0·11719 | 0·12414 |
| 0·60 | 0·11387 | 0·5 | 0·13437 | 0·17484 | 0·14341 | 0·12904 | 0·12198 |
| 0·80 | 0·30576 | 0·5 | 0·30277 | 0·37943 | 0·33713 | 0·32215 | 0·31492 |
| 1·00 | 0·43171 | 0·5 | 0·41667 | 0·51566 | 0·46570 | 0·44985 | 0·44228 |

$k$ = iteration number.

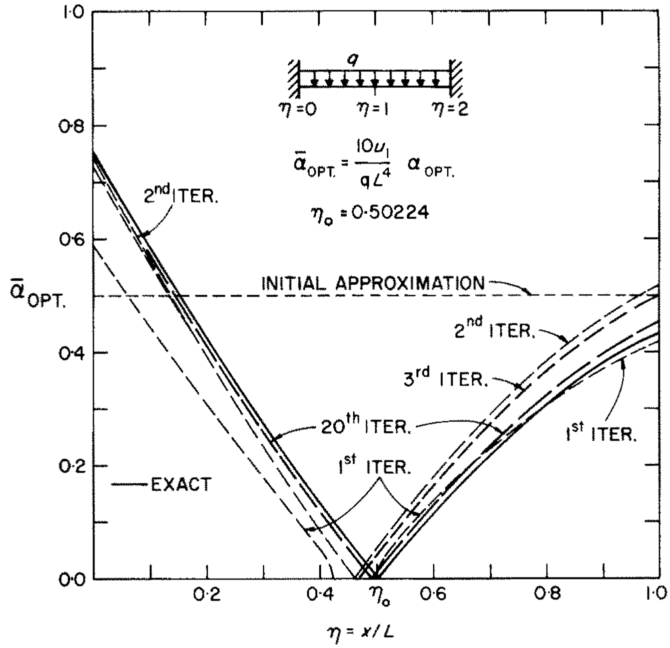


FIG. 5. Comparison of successive approximation with optimal design in a clamped beam under uniformly distributed load.

method of successive approximation is similar to that presented in Section 6, making $k = 0$ and where the Riccati equations (25) and (29) were subject to appropriate initial conditions in order to account for the difference in boundary conditions of the present beam.

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**Абстракт**—В настоящей работе, путем использования вариационного исчисления, получаются условия оптимальности для балки на упругом основании, подверженной действию ограниченного перемещения. Впоследствие, эти условия используются в смысле точки начала для определения стабильного метода для численного решения задачи оптимизации. На основе идеи инвариантной заделки и метода постепенных приближений, сводится принадлежащая нелинейная краевая задача к двум шаблонам итеративного процесса, в форме системы дифференциальных уравнений Рикатти, при заданных начальных условиях, которые проявляют удобные свойства устойчивости. Даются два примеры, разработаны до конца, с целью иллюстрации применимости и точности метода.